LEBANESE AMERICAN UNIVERSITY
Department of Computer Science and Mathematics

Calculus III

Exam I

Fall 2015 (Oct 6, 2015)

Exam duration: 75 minutes

Name: Solutions ID: ______

QUESTION	GRADE
1. 10 %	
2. 10%	
3. 56%	
4. 9%	
5. 15%	
TOTAL	

1. (10%) Integrate the following

(a) $\int \frac{x\,dx}{1+x^4}$ $\quad u = x^2 \quad du = 2x\,dx.$

$= \frac{1}{2}\int \frac{du}{1+u^2} = \frac{1}{2}\tan^{-1}(x^2) + c$

(b) $\int \frac{4x-5}{x^3-3x^2}dx \quad = \int \frac{4x-5}{x^2(x-3)}\,dx.$

$\frac{4x-5}{x^2(x-3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-3} \Rightarrow Ax(x-3) + B(x-3) + Cx^2 = 4x-5$

$\Rightarrow \underbrace{(A+C)}_{0}x^2 + (\underbrace{-3A+B}_{4})x - \underbrace{3B}_{-5} = 4x-5.$

$B = 5/3$

$-3A = 4 - B$

$A = -\frac{4}{3} + \frac{5}{9}$

$C = \frac{4}{3} - \frac{5}{9}$

$\therefore$ Int. $= A\ln|x| - \frac{B}{x} + C\ln|x-3| + k.$

2. (10%) Evaluate the following (improper) integrals:

(a) $\int_0^\infty \frac{2e^x}{1+e^{2x}}dx \quad = \quad u = e^x : \quad \int \frac{2\,du}{1+u^2}.$

$= \lim_{t\to\infty} 2\tan^{-1}(e^x)\Big|_0^t = 2\tan^{-1}(\infty) - 2\tan^{-1}(1)$

$= 2(\pi/2) - 2(\pi/4) = \pi - \pi/2 = \pi/2$

$\therefore$ conv. to $\pi/2$

(b) $\int_8^\infty \frac{dx}{(x-2)(x+3)}$ Partial fraction.

$$\frac{1}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3} \qquad A = 1/5 \qquad B = -1/5.$$

$$= \lim_{t\to\infty} \frac{1}{5} \ln\left(\frac{x-2}{x+3}\right)\Big|_8^t \longrightarrow \boxed{-\frac{1}{5}\ln(6/18)}$$

3. (56%) Determine whether the following improper integrals converge or diverge:

(a) $\int_4^5 \frac{dx}{\sqrt{|x-4|}}$ $= \lim_{t\to 4^+} \int_t^5 \frac{dx}{\sqrt{|x-4|}}$

$$= \lim_{t\to 4^+} \left(2\sqrt{x-4}\Big|_t^5\right) \to 2 \Rightarrow \underline{\text{converge}}$$

(b) $\int_0^\infty \frac{\sqrt{x}}{1+x^2}dx$ $= \int_0^1 \frac{\sqrt{x}}{1+x^2}dx + \int_1^\infty \frac{\sqrt{x}}{1+x^2}dx$

proper ⇒ bounded

$$\approx \int_1^\infty \frac{dx}{x^{1.5}}$$

conv.
p-int. $p > 1$.

$\therefore$ $\underline{\text{converges.}}$

(c) $\int_1^\infty \frac{\ln x}{(x+e^x)}dx$ $\approx \int_1^\infty \frac{\ln x}{e^x}\,dx$

$\ln x < x$

$e^x > x^3 \Rightarrow \frac{1}{e^x} < \frac{1}{x^3}$ $\therefore$ $\frac{\ln x}{e^x} < \frac{x}{x^3} = \frac{1}{x^2}$.

$\therefore \int_1^\infty \frac{\ln x}{e^x}\,dx < \int_1^\infty \frac{1}{x^2}\,dx$ $\therefore$ converges by DCT

(d) $\int_1^\infty \frac{e^x}{\sqrt{1+x^2}}dx$

$\frac{e^x}{\sqrt{1+x^2}} \longrightarrow \infty$ $\therefore$ $\int_1^\infty \frac{e^x}{\sqrt{1+x^2}}\,dx \longrightarrow \boxed{\infty}$

diverges

(e) $\int_{-\infty}^\infty \frac{1}{\sinh x}dx$ odd fn

$\therefore$ consider $\int_0^\infty \frac{dx}{\sinh x} = \int_0^\infty \frac{2}{e^x - e^{-x}} \overset{LCT}{\approx} \int_0^\infty \frac{2}{e^x}$

$\therefore$ converges $\boxed{\text{to } L}$ since $e^x > x^2$ (or simply integrate).

conclusion. the whole integral converges to $L - L = 0$

(f) $\int_0^\infty \frac{|\sin x|}{1+x^2}dx$ $= \underbrace{\int_0^1 \frac{|\sin x|}{1+x^2}dx}_{\text{proper int.} \Rightarrow \text{bdd}} + \int_1^\infty \frac{|\sin x|}{1+x^2}dx$

$$\int_1^\infty \frac{|\sin x|}{1+x^2}dx < \int_1^\infty \frac{1}{1+x^2}dx \approx \int_1^\infty \frac{dx}{x^2}$$: p-int. $p>1$ $\Rightarrow$ conv.

$\therefore$ The original int = (proper int + conv. int.) $\Rightarrow$ converges

(g) $\int_1^\infty x\sin(1/x)dx$

Note that $x\sin(1/x) = \left(\frac{\sin 1/x}{1/x}\right) \xrightarrow[x\to\infty]{} 1$

$\therefore$ integral $\overset{\text{LCT}}{\approx} \int_1^\infty 1\,dx = \infty$ $\therefore$ diverges

(h) $\int_8^\infty \frac{x^{1.99}dx}{(x-2)(x+3)}$ $\approx \int_8^\infty \frac{x^{1.99}}{x^2}dx = \int_8^\infty \frac{dx}{x^{0.01}}$

p-int. $p<1$
$\Rightarrow$ diverges.

4. (9%) Find the value of $p > 0$ for which the integral $\int_2^\infty \frac{\ln x}{x^p}\,dx$ converges.

9 if no clear reason

Case 1 If $p \leq 1 \Rightarrow$ Use $\ln x > 1$ for (x large) $\Rightarrow$ int. $> \int \frac{1}{x^p}$ $p<1$ $\Rightarrow$ div.

Case 2 If $p > 1 \Rightarrow$ Use $\ln x < x^{\varepsilon}$ ε small enough and positive so that $p - \varepsilon$ still > 1

$\therefore$ $\boxed{\text{conv. for } p > 1}$

5. (15%) Consider the following sequences. Determine if they converge or diverge,. In case if convergence find their limit:

(a) $a_n = (2n+1)^{1/n^2}$

$\approx (2n)^{1/n^2} = \left[(2n)^{1/n}\right]^{1/n} \longrightarrow 1^{1/n} \longrightarrow 1$

conv. to 1

(b) $a_n = \left(\frac{2n+3}{2n}\right)^n = \left(1 + \frac{3}{2n}\right)^n = \left(1 + \frac{3/2}{n}\right)^n \longrightarrow e^{3/2}$

conv. to $e^{3/2}$

(c) $a_n = (-1)^n\left(1 - \frac{1}{n}\right) \longrightarrow (-1)^n(1) \Rightarrow$ diverges (oscillates b/w $1, -1$)